Name: ______________________ Date: ____________ Period: ________

4.10 Polynomial Toolkit

Vocabulary and Theorems

A **polynomial** function is of the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_1 x + a_0$$

where $a_n, a_{n-1}, \dots a_0$ are real **coefficients** ($a_n \neq 0$) and n is a non-negative integer.

The **degree** of the polynomial is n. The **leading coefficient** is a_n. The **constant term** is a_0.

The formal classifications of polynomials are given below.

Classification by Degree		Classification by Number of Terms	
0	Constant		
1	Linear	1	Monomial
2	Quadratic	2	Binomial
3	Cubic	3	Trinomial
4	Quartic	4+	Multinomial or Polynomial
5	Quintic		
6+	Nth degree		

Degree: the biggest exponent in a single term for the entire expression. (If there are multiple variables in a term: add up the exponents for all variables in that term, then determine the biggest exponent).

Standard form of a polynomial: putting the terms in order from the biggest exponent to the smallest exponent.

Dividing form of a polynomial: putting the terms in order from biggest exponent to the smallest exponent and including 'zero terms' (terms with zero coefficients) that are not written in standard form.

For example, $f(x) = x^3 - 2x$ is a cubic binomial in standard form. Its dividing form would be: $f(x) = x^3 + 0x^2 - 2x + 0$ and its factored form would be: $f(x) = x(x^2 - 2)$.

Remember:	should have:
a Quartic (**degree 4**)	5 terms
a Cubic (**degree 3**)	4 terms
a Quadratic (**degree 2**)	3 terms
a polynomial of **degree n**	n + 1 terms

The graph of a polynomial function is **continuous** and has only smooth rounded turns.

The graph of a polynomial function can have at most n – 1 turning points, also known as **extrema** (relative minimums and relative maximums).

Factor Theorem: A linear binomial $mx + b$ is a factor of another polynomial if the remainder is zero when you divide by the linear binomial.

Remainder Theorem: If a polynomial, $f(x)$, is divided by $x - a$, then the remainder is equal to $f(a)$.

Complex Conjugate Theorem: $a + bi$ is a root of the polynomial if and only if $a - bi$ is a root. Implication: imaginary and complex roots always come in conjugate pairs.

Linear Factorization Theorem: If $f(x)$ is a polynomial function of degree n, where n > 0, then f has precisely n linear factors and $f(x)$ can be written as: $\boldsymbol{f(x) = a_n(x - c_1)(x - c_2) \dots (x - c_n)}$ where $c_1, c_2, \dots c_n$ are complex numbers.

As $x \to -\infty$ Left side | As $x \to \infty$ Right side

End Behavior

Note

The first term of a polynomial expression, $a_n x^n$, in standard form is very informative. It tells you the degree of the polynomial (n) and the leading coefficient helps to determine the **end behavior** of the polynomial. The leading coefficient test is described below.

The Leading Coefficient Test

		LEADING COEFFICIENT	
		POSITIVE	**NEGATIVE**
DEGREE	**ODD**	As $x \to -\infty, y \to -\infty$ As $x \to \infty, y \to \infty$ $\swarrow \quad \nearrow$	As $x \to -\infty, y \to \infty$ As $x \to \infty, y \to -\infty$ $\nwarrow \quad \searrow$
	EVEN	As $x \to -\infty, y \to \infty$ As $x \to \infty, y \to \infty$ $\nwarrow \quad \nearrow$	As $x \to -\infty, y \to -\infty$ As $x \to \infty, y \to -\infty$ $\swarrow \quad \searrow$

DG LC

Based on the degree and leading coefficient, find the end behavior for the polynomial equations below:

1] $f(x) = 3x^3 - 5x + 2x - 6$ LC = 3 (positive) DG = 3 (odd) As $x \to -\infty, y \to -\infty$ As $x \to \infty, y \to \infty$	**2]** $g(x) = -5x^4 + 21$ LC = -5 (negative) DG = 4 (even) As $x \to -\infty, y \to -\infty$ As $x \to \infty, y \to -\infty$
3] $h(x) = -x^5 + 4x^4 - 10x^3 + 5x^2 + 3$ LC = -1 (negative) DG = 5 (odd) As $x \to -\infty, y \to \infty$ As $x \to \infty, y \to -\infty$	**4]** $m(x) = 24x^2 - 15x + 30$ LC = 24 (positive) DG = 2 (even) As $x \to -\infty, y \to \infty$ As $x \to \infty, y \to \infty$

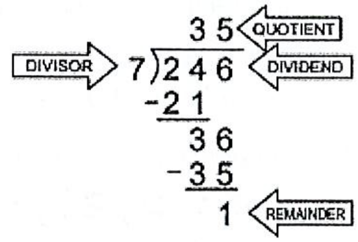
End Behavior Jig

!DISCO! | TOUCHDOWN | MUSCLE MAN

POS & ODD | NEG & ODD | POS E-VEN | NEG E-VEN

Long Division

When you want to divide a polynomial by another polynomial, you can use a process called long division as long as the degree of the numerator is larger than the degree of the denominator. It is similar to the process for numbers (see example below).



1) Set up the division problem so the dividend and divisor are written in dividing form.

2) Divide the leading term of the dividend by the leading term of the divisor. This will be the first term of your quotient.

3) Multiply the term times the divisor and put it under the dividend.

4) Subtract. (The first term should go away and bring down the next term(s) so that you have the same number of terms as the divisor). This is your modified dividend. *Remember subtracting requires you to change the signs of the second polynomial then add/combine like terms.*

5) Repeat the pattern of steps 2-5 until there are no more terms to bring down. The last line will be the **remainder**.

6) Write the answer. Remember $\textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{Divisor}}$ **Note:** If the remainder is zero, then the divisor is a factor of the dividend.

5] Divide $12x^3 - 11x^2 + 9x + 18$ by $4x + 3$

$$\boxed{3x^2 - 5x + 6}$$

$$4x+3 \overline{)\, 12x^3 - 11x^2 + 9x + 18}$$

$$-(\cancel{12x^3} + 9x^2) \downarrow$$

$$-20x^2 + 9x$$

$$-(-\cancel{20x^2} - 15x) \downarrow$$

$$24x + 18$$

$$-(24x + 18)$$

$$0 \leftarrow \text{remainder}$$

$\frac{12x^3}{4x} = 3x^2$

$\frac{-20x^2}{4x} = -5x$

$\frac{24x}{4x} = 6$

special Note
remainder = 0 means the divisor is a FACTOR of the dividend.

$$12x^3 - 11x^2 + 9x + 18 = (4x+3)(3x^2 - 5x + 6)$$

6] Divide using long division.

$$\frac{4x^3 - 2x^2 - 3}{2x^2 - 1}$$

dividing form: $\frac{4x^3 - 2x^2 + 0x - 3}{2x^2 + 0x - 1}$

$$2x - 1$$

$$2x^2 + 0x - 1 \overline{)\, 4x^3 - 2x^2 + 0x - 3}$$

$$-(\cancel{4x^3} + 0x^2 - 2x) \downarrow$$

$$-2x^2 + 2x - 3$$

$$-(-\cancel{2x^2} + 0x + 1)$$

$$2x - 4 \text{ remainder}$$

$\frac{4x^3}{2x^2} = 2x$

$\frac{-2x^2}{2x^2} = -1$

$$\boxed{2x - 1 + \frac{2x-4}{2x^2-1}}$$

Standard Form from Zeros

1) List the zeros and any implied zeros from Complex Conjugate Theorem (if applicable).

2) Write the linear factors using the Linear Factorization Theorem.

3) Multiply the factors together carefully. Use the DOS/SOS/PST identities as necessary.

4) If given a point, use substitution to find the leading coefficient, a, otherwise choose your own value for a. $a \neq 0$.

7] Find a cubic polynomial function that has zeros 2 and $3i$. Write in standard form.

zeros: $2, 3i, -3i$ ← implied [conjugate of $3i$]

choose $a=1$

$y = a(x-2)\underbrace{(x-3i)(x+3i)}_{\text{SOS}}$

SOS: $(a+bi)(a-bi) = a^2+b^2$

$y = 1(x-2)(x^2+9)$

FOIL

$y = x^3+9x-2x^2-18$

$\boxed{y = x^3-2x^2+9x-18}$

8] Find a polynomial function of degree 4 that has zeros, 0, 2 and $1+i$ with a leading coefficient of -2.

Zeros: $0, 2, 1+i, 1-i$ ← implied [conjugate of $1+i$]

leading coeff.

$y = -2(x)(x-2)[x-(1+i)][x-(1-i)]$

	x	-1	$-i$
x			
-1			
$+i$			

$y = -2x(x-2)\underbrace{[(x-1)-i][(x-1)+i]}_{\text{SOS}}$

$y = -2x(x-2)[\underbrace{(x-1)^2}_{\text{PST}}+1]$

PST:
$(a-b)^2 = a^2-2ab+b^2$

$y = (-2x^2+4x)[x^2-2x+1+1]$

$y = (-2x^2+4x)(x^2-2x+2)$

We will use this toolkit to help us with solving Polynomial Puzzles in the next lesson.

$y = -2x^4+8x^3-12x^2+8x$

Standard form

	x^2	$-2x$	$+2$
$-2x^2$	$-2x^4$	$4x^3$	$-4x^2$
$4x$	$4x^3$	$-8x^2$	$8x$

$y = -2x(x-2)(x^2-2x+2)$

factored form