Name: ______________________________ Date: __________ Period: ________

2.6 **Café Mathematics**

Explore: Café Mathematics

John Napier is running late in meeting with his friend Leonhard Euler at a local coffee shop, The Café Mathematica. John is behind schedule because he has spent all morning making a discovery, namely that:

$$f(x) = \log_b x \text{ has some relation to } x = b^{f(x)}$$

While Leonhard is waiting for his hopelessly tardy friend, he begins scribbling out the solutions to some exponential problems that have been posed by a few of his professor friends at the local university on some spare napkins at the table.

Part One

1) The population of a town increases according to the model:

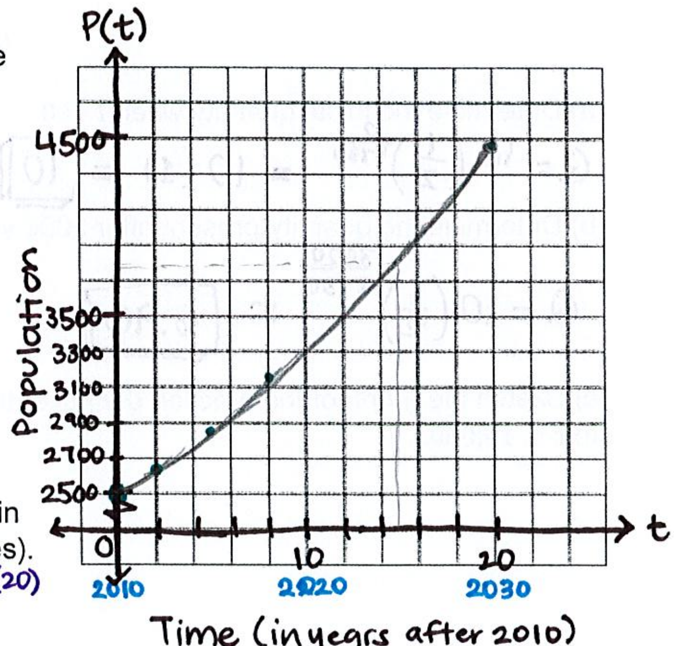
$$P(t) = 2500e^{0.0293t}$$

where t is the time in years, with $t = 0$ corresponding to 2010.

a) Find the projected population of the town in 2012, 2015, and 2018.

Year	2012	2015	2018
t	2	5	8
P(t)	2650	2894	3160

b) Use a graphing calculator to graph the function for the years 2010 through 2030 on the coordinate plane to the right.



c) Use ~~a graphing calculator~~ the graph to ***approximate*** the population in 2025 and 2030.

Year	2025	2030
t	15	20
P(t)	3800	4500

d) Verify your answers in part c) algebraically (answers in exact form and approximate form to three decimal places).

$P(15) = 2500e^{.0293(15)}$

$= 2500e^{0.4395}$ exact

≈ 3879.828 approx.

$P(20) = 2500e^{.0293(20)}$

$= 2500e^{.586}$ exact

≈ 4491.967 approx.

2) A certain population increases according to the model $P(t) = 250e^{0.47t}$. Use the model to determine the population when $t = 5$. Round your answer to the nearest integer.

A) 400
B) 2621
C) 1998
D) 1596
E) None of these

$P(5) = 250e^{.047(5)}$

$\boxed{P(5) = 2621.392...}$

d	day
0	1st
1	2nd

3) You go to work for a company that pays \$0.01 the first day, \$0.02 the second day, \$0.04 the third day, and so on. If the daily wage keeps doubling, what will your total income be after working 15 days? $y = 0.01(2)^d$

A) \$0.15
B) \$2.02
C) \$32.00
D) \$327.67
E) \$32 767.00

15th day $y = 0.01(2)^{15}$

$\boxed{y = 327.68}$

4) You bought a guitar 6 years ago for \$400. If its value decreases by about 13% per year, how much is your guitar worth now?

A) \$173.45
B) \$226.55
C) \$322.00
D) \$351.23
E) \$832.78

Model $y = 400(1-0.13)^t$ t = # of years

$y = 400(0.87)^6$ $\boxed{\$173.45}$

5) The amount of a certain radioactive substance remaining after t years decreases according to the function $N = N_0 e^{-0.0315t}$ where N_0 is the initial amount of the substance and t = time in years. How much of a $\underbrace{\text{25-gram}}_{N_0}$ sample will remain after $\underbrace{\text{20 years}}_{t}$?

A) 13.31 grams
B) 46.94 grams
C) 0.53 grams
D) 1.88 grams

$N = 25e^{-0.0315(20)}$

$\boxed{N = 13.3147...}$

6) Let Q (in grams) represent the mass of a quantity of carbon-14, which has a half-life of 5730 years. The quantity present after t years is:

$$Q = 10\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

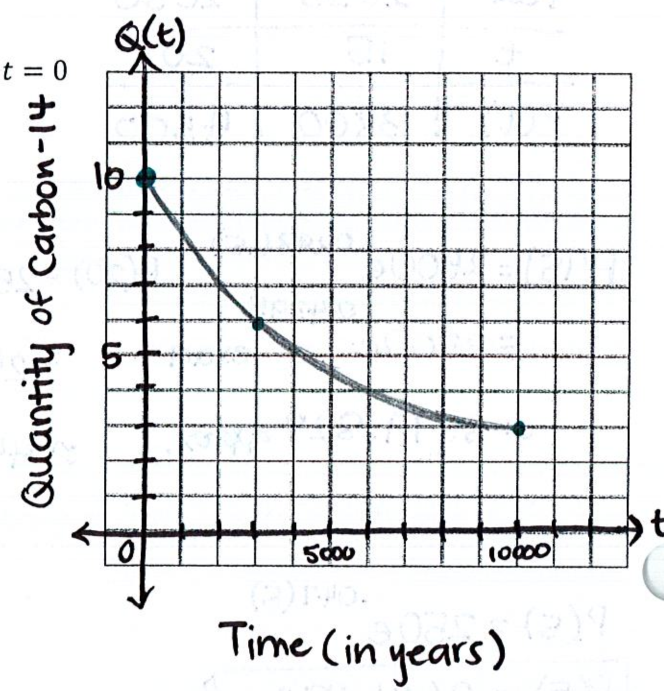
a) Determine the initial quantity when $t = 0$.

$Q = 10\left(\frac{1}{2}\right)^{\frac{0}{5730}} = 10(1) = \boxed{10}$

b) Determine the quantity present after 3000 years.

$Q = 10\left(\frac{1}{2}\right)^{\frac{3000}{5730}} \approx \boxed{6.96}$

c) Sketch the graph of the function $Q(t)$ over the interval $t = 0$ to $t = 10{,}000$.



Part Two

Now that John has shown up at The Café Mathematica, he wishes to share his new knowledge with his buddy Leonhard. John offers Leonhard the following problems to work out and discuss, so Leonhard immediately asks for new napkins on which to scribble profusely. Up until John's discovery, the two couldn't figure these problems out!

7) The value of a snowmobile can be modeled by the equation $y = 4500(0.93)^t$ where t is the number of years since the car was purchased. After how many years will the value of the snowmobile be about $2500?

A) 7 years
B) 8 years
C) 9 years
D) 10 years

$\frac{2500}{4500} = \frac{4500(0.93)^t}{4500}$

$\frac{5}{9} = (0.93)^t$

$\boxed{t = \log_{0.93}\left(\frac{5}{9}\right)}$ exact

$\boxed{t \approx 8.099\ldots}$ approx.

8) The amount of a certain radioactive substance remaining after t years decreases according to the function $N = N_0e^{-0.0315t}$ where N_0 is the initial amount of the substance and t = time in years. Approximately how many years will it take for a 30-gram sample to decay to 15 grams?

A) -22 years
B) 22 years
C) 18.70 years
D) 5.83 years

$\frac{15}{30} = \frac{30e^{-0.0315t}}{30}$

$0.5 = e^{-0.0315t}$

$\frac{-0.0315t}{-0.0315} = \frac{\ln 0.5}{-0.0315}$

$\boxed{t = -\frac{1}{0.0315}\ln 0.5}$ exact

$\boxed{t \approx 22.004\ldots}$ approx.

9) The formula for finding the number of bacteria present is given by $P = P_0(2)^{2t}$ where P is the final population, P_0 is the initial population and t is the time measured in hours. If the population contained 275 bacteria at t = 0, approximately how long will it take for 15,000 bacteria to be present?

A) 2.25 hours
B) -2.88 hours
C) -2.25 hours
D) 2.88 hours

$\frac{15000}{275} = \frac{275(2)^{2t}}{275}$

$\frac{600}{11} = 2^{2t}$

$\frac{2t}{2} = \frac{\log_2\left(\frac{600}{11}\right)}{2}$

$\boxed{t = \frac{1}{2}\log_2\left(\frac{600}{11}\right)}$ exact

$\boxed{t \approx 2.8846\ldots}$ approx.

10) After John shares his excitement over his newly-discovered logarithms, Leonhard decides to change the subject (apparently Euler was a very jealous mathematician) to an investment opportunity.

Leonhard has an investment opportunity for John that will pay John 8.73% interest compounded annually if John makes an initial investment of $50,000.

$r = 0.0873$ $n = 1$ $P = 50000$

model: $A = 50000\left(1 + \frac{.0873}{1}\right)^{1(t)}$

a) How long will it take for John to double his money?

$\frac{100000}{50000} = \frac{50000(1.0873)^t}{50000}$

$2 = 1.0873^t$

$\boxed{t = \log_{1.0873}(2) \approx 8.28 \text{ years}}$

exact approx.

b) How long will it take for John's investment to have a value of $68,000?

$\frac{68000}{50000} = \frac{50000(1.0873)^t}{50000}$

$\frac{34}{25} = 1.0873^t$

$\boxed{t = \log_{1.0873}\left(\frac{34}{25}\right)}$ exact

$\boxed{t \approx 3.67 \text{ years}}$ approx

c) Leonhard is being dishonest with John. The investment actually pays *8.73% interest compounded continually*. Leonhard, who is extremely jealous that John has discovered logarithms, plans on keeping the extra interest for himself when John is paid 8.73% interest compounded annually. After five years, how much money would Leonhard make off of cheating John?

compounded annually

$A = 50000(1.0873)^5$

$A = 75983.09$

compounded continuously

$A = 50000e^{.0873(5)}$

$A = 77364.11$

$77364.11 - 75983.09$

$= \boxed{\$1381.02}$

d) Luckily, John doesn't fall for Leonhard's investment trick. John tells Leonhard that he would only invest $50,000 in an investment that would double in five years. If interest was being compounded continuously, what interest rate would John need to do this?

$\frac{100000}{50000} = \frac{50000e^{5r}}{50000}$

$2 = e^{5r}$

$\frac{5r}{5} = \frac{\ln 2}{5}$

$\longrightarrow r = \frac{\ln 2}{5}$ exact

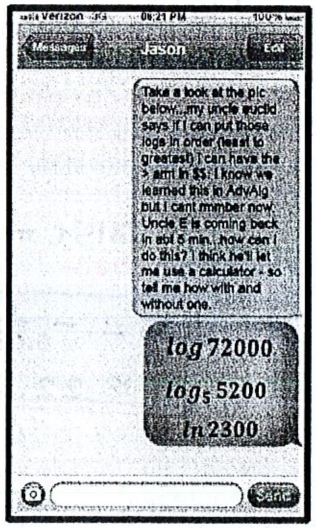
$r \approx 0.1386\ldots$ approx

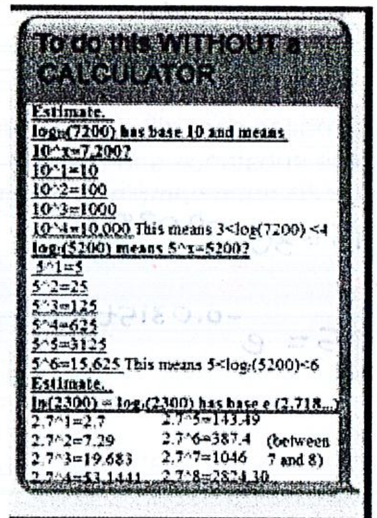
The interest rate needs to be about 13.9%.

Seized with guilt, Leonhard breaks into tears and confesses his scheme to John. John hugs Leonhard and tells him that he forgives him. The two walk out of The Café Mathematics together, and the other customers couldn't be happier. They were afraid of the two crazy men at the table scribbling things on napkins and arguing about lumber. And they didn't even order anything.

Apply: Jason's Graduation Present

1) You are at a baseball game when you get a text from your friend Jason. See to the left. Jason's uncle, an eccentric mathematics teacher, has decided to make Jason's graduation gift a challenge. You have to text him back the instructions for how to evaluate and order those logarithms without a calculator. Finish the texts below and help Jason.





$\boxed{\log 7200 < \log_5 5200 < \ln 2300}$

2) Jason received $1000 in graduation gifts. He found a savings plan that will pay him 4% interest compounded continuously. Use the continuously compounded interest formula to write the amount of money *A* Jason will have as a function of the time in *t* years.

$P = 1000$ $\quad r = .04$

$\boxed{A(t) = 1000e^{.04t}}$

3) Jason wants to know how long it will take for him to have $1500 in his account. Calculate this time and round to three significant figures.

$\frac{1500}{1000} = \frac{1000e^{.04t}}{1000}$

$1.5 = e^{.04t}$

$\frac{.04t}{.04} = \frac{\ln 1.5}{.04}$

$\boxed{t = \frac{1}{0.04}\ln 1.5 \text{ or } 25\ln 1.5}$ exact

$t \approx 10.1$ years approx

4) One of Jason's graduation gifts is a trip on Amtrak from his home in Atlanta to New York to visit a cousin. Jason is afraid he may experience motion sickness so he has decided to take 50 mg of dimenhydrinate (which will help prevent motion sickness) before boarding Amtrak. Suppose that 85% of this medication remains in the bloodstream after 1 hour. Represent the amount of dimenhydrinate in Jason's bloodstream for the first 4 hours of taking the drug. Let *t* be the number of hours after reaching its peak level of 50 mg.

t (in hours)	0	1	2	3	4
amount of dimenhydrinate	50	42.5	36.125	30.706	26.1

5) Write an exponential function for the amount of dimenhydrinate in Jason's bloodstream t hours after reaching its peak level of 50 mg. How long does it take for Jason to have only half the amount of dimenhydrinate in his bloodstream? Determine the solution and explain your thinking.

function: $y = 50(0.85)^t$

$\frac{25}{50} = \frac{50(0.85)^t}{50}$

$0.5 = 0.85^t$

$\boxed{t = \log_{0.85}(0.5)}$ exact

$t \approx 4.26$ hours approx

We knew it would be a little longer than 4 hours since amt was 26.1 at $t = 4$.

6) At what level might you consider Jason's bloodstream cleared of dimenhydrinate? Why did you choose this level? Using your established criteria, find how long it takes for the dimenhydrinate to clear Jason's bloodstream. Explain how you found your solution. Answers will vary.

Sample answer: I would choose 0.01 mg as cleared from the system since most measurements are taken ~~to~~ the nearest tenth of a gram.

$\frac{0.01}{50} = \frac{50(0.85)^t}{50} \Longrightarrow 0.0002 = 0.85^t \Longrightarrow \boxed{t = \log_{0.85}(0.0002) \approx 52.4 \text{ hours}}$

$\log_{0.85}(0.0002)$ exact, 52.4 hours approx